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Probing integrable perturbations of conformal theories using singular vectors II: $N=1$ superconformal theories

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ABSTRACT

In this work we pursue the singular-vector analysis of the integrable perturbations of conformal theories that was initiated in [1]. Here we consider the detailed study of the $N = 1$ superconformal theory and show that all integrable perturbations can be identified from a simple singular-vector argument. We identify these perturbations as theories based on affine Lie superalgebras and show that the results we obtain relating two perturbations can be understood by the extension of affine Toda duality to these theories with fermions. We also discuss how this duality is broken in specific cases.

1 Introduction

An integrable perturbation of a conformal field theory is defined to be one which has a sufficient number of conserved quantities, which for most cases means a single nontrivial (conformal dimension > 1) integral of motion. An integral of motion is conserved in the perturbed theory if the single-pole term in the OPE of the conserved density with the perturbing field is a total derivative. So far the perturbations which are known to be integrable fall (apart from a few cases) into series valid for all c , and have been shown to be integrable using either Zamolodchikov's counting argument [2] or Feigin and Frenkel's cohomological argument based on affine quantum groups [3].

In ref [1], we showed that the integrable perturbations of the conformal minimal models can also all be identified by means of a rather simple singular-vector analysis. We take a perturbation $\phi_{r,s}(z)\bar{\phi}_{r,s}(\bar{z})$, in a (p, p') minimal model with central charge

$$c(p, p') = 1 - 6 \frac{(p - p')^2}{pp'} . \quad (1.1)$$

The holomorphic part of the perturbation has conformal dimension $h_{r,s}$ given by:

$$h_{r,s} = \frac{(rp - sp')^2 - (p - p')^2}{4pp'} \quad 1 \leq r \leq p' - 1, 1 \leq s \leq p - 1 . \quad (1.2)$$

This field has primitive singular vectors at level rs and $(p' - r)(p - s)$. All other singular vectors are descendant of these two. We specialise ourself to the 'space of fields' $\mathcal{F}_{r,s}$, in which the singular vector at level rs of each primary field is identically set equal to zero. As a result, all 'secondary' singular vectors disappear, leaving only the primitive singular vector at level $(p' - r)(p - s)$.

The point of the singular-vector analysis of integrable perturbations is the following: the existence of a conserved density can be easily unravelled if, for some value of the central charge, this density is not only a vacuum descendant but also a primary field, hence necessarily singular.¹ Denote such densities as χ_I (where here and below, χ_ϕ denotes the degenerate field associated to the singular vector of the module $|\phi\rangle$ in the quotient space \mathcal{F}). The OPE under consideration is thus of the form

$$P \times \chi_I = \chi_P , \quad (1.3)$$

where P stands for the (holomorphic part of the) perturbing field. A total derivative pole term will automatically appear if the difference

$$\Delta h = h_p + h_{\chi_I} - h_{\chi_P} , \quad (1.4)$$

(which is here the difference between the level of the vacuum and the perturbed singular

¹That conserved densities can be singular at particular values of c was first noticed in [4] – see also [5, 6].

vectors) is 2: indeed, the OPE is then

$$P(z)\chi_I(w) = \frac{1}{(z-w)^2}[\chi_P(w) + (z-w)k\partial\chi_P(w)] , \quad (1.5)$$

where k is a constant. In the Virasoro case, explicit examination of opes with higher order poles shows it is very unlikely that they will give conserved quantities.² Therefore, putting in light nontrivial (albeit singular) conservation laws in a given perturbed theory boils down to analysing the solutions of the condition $\Delta h = 2$. Since the level of the vacuum singular vector is $(p' - 1)(p - 1)$ and that of the perturbing vector of type (r, s) is $(p' - r)(p - s)$, we have

$$\Delta h = (p' - r)(s - 1) + (p - s)(r - 1) + (r - 1)(s - 1) = 2 . \quad (1.6)$$

Having written the condition in terms of a sum of positive terms, it is then clear that there is a very limited number of solutions. One of r or s is necessarily 1. Setting $r = 1$ yields

$$(p' - 1)(s - 1) = 2 \quad \Rightarrow \quad p' = 2, s = 3 \quad \text{or} \quad p' = 3, s = 2 . \quad (1.7)$$

The first case corresponds to the perturbation $\phi_{1,3}$. When $p' = 2$, p is forced to be odd, say $p = 2k + 1$. The dimension of the singular conserved densities (integrals) are then $2k$ (resp. $2k - 1$). The second solution establishes the integrability of the perturbation $\phi_{1,2}$, in which case the singular conserved densities have dimension $6k \pm 2$. It is worth emphasising that in each case, an infinite number of conservation laws is obtained, with dimensions varying with the central charge. By duality [3], $\phi_{3,1}$ and $\phi_{5,1}$ are also integrable perturbations, and consequently, by interchanging the role of p and p' , $\phi_{1,5}$ is integrable.

In the present paper, we extend our analysis to the minimal models of the $N = 1$ superconformal algebra, whose structure is briefly reviewed in section 2. So far three integrable perturbations have already been identified: $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ [7], which preserves the supersymmetry invariance off criticality, and $\hat{\phi}_{1,3}$ [8] and $\hat{\phi}_{1,5}$ [9] which both break supersymmetry. These three perturbations can be identified as affine Toda theories based on Lie superalgebras through the free-field construction, which we present in section 6. As a result, they should exhibit ‘duality’ inherited from the free-field construction, as noted implicitly in [10] and observed in the S-matrices [11]. In our case this means that we can identify each perturbation with an affine super algebra g and coupling constant β , and that the theories (g, β) and $(g^\vee, -1/\beta)$ (where g^\vee is the dual algebra to g) should be equivalent.

The singular-vector analysis of these models is presented in sect. 3. It reveals several new features which were not seen in [1]:

i) The first obvious source of novelties is related to the presence of fermionic fields. This leads to possible fermionic integrals of motion. We indeed find that all singular conservation laws for the $\hat{\phi}_{1,5}$ perturbation are fermionic. As already mentioned in the introduction, the presence of a nontrivial conservation law is generally taken as a reliable integrability indicator. But this

² Moreover, if there are solutions corresponding to the cases $\Delta h > 2$, these cannot be generic, i.e., they do not hold for an infinite sequence of models (in contrast to the situation for $\Delta h = 2$ – see below). In that sense, they would not be viewed as a genuine signal of integrability.

is only so for a bosonic conservation law. The situation appears to be quite different when the nontrivial conservation law is fermionic. In that case, even the presence of an infinite number of fermionic conserved quantities is not sufficient for integrability. We illustrate this point in appendix A by constructing classical evolution equations for a bosonic field interacting with a fermionic field (not necessarily invariant under supersymmetry) that display an infinite number of fermionic conservation laws without being integrable.

ii) For the $\hat{\phi}_{1,5}$ perturbation, not only are all the predicted singular conserved quantities fermionic, but these quantities, which are predicted at specific values of c , do not extend away from these c -values. This is in sharp contrast with all other cases studied previously: whenever an infinite number of (singular) conservation laws (with dimensions varying with c) is found, these always proved to be particular cases of genuine conservation laws that exist for generic values of c . In subsection 8.3, we relate these virtual fermionic conservation laws to the presence of a (severely truncated) $WB(0, m)$ symmetry at the particular values of c where they appear, the conserved quantities being the remnants of the fermionic generator of this algebra.

iii) The present analysis also uncovers a surprising breakdown in duality: the singular conserved densities in the model (g, β) are not always equal to those in $(g^\vee, -1/\beta)$. A free-field construction explanation is worked out in sections 6 and 7.

iv) Finally we observe systematic enhancements in the number of conservation laws of a particular perturbation for certain values of c . This is again explained in general terms from the free-field construction in sections 6 and 7, and a more precise analysis pertaining to certain cases is presented in section 8.

We stress that in spite of the ambiguous signals related to points (i) and (ii), the $\hat{\phi}_{1,5}$ perturbation is integrable and in fact it has exactly the same **generic** conserved quantities as the $\hat{\phi}_{1,3}$ perturbation – up to p, p' interchange – although none of these bosonic conserved densities ever become singular. The duality relating the two supersymmetry-breaking perturbations is explained in section 5.

2 $N = 1$ superconformal models

We start by reviewing the structure of the highest-weight representations of the $N = 1$ superconformal algebra and their singular vectors. The Verma modules \hat{M} are labelled by c and h which we parametrise as³

$$c = \hat{c}(t) = \frac{15}{2} - \frac{3}{t} - 3t, \quad h = \hat{h}_{r,s}(t) = \frac{r^2 - 1}{8t} + \frac{(s^2 - 1)t}{8} + \frac{1 - rs}{4} + \frac{1 - (-1)^{r+s}}{32}. \quad (2.1)$$

We shall usually denote a highest-weight vector (whether the highest-weight vector of smallest \hat{h} in a Fock or Verma module, or an embedded highest-weight vector) corresponding to these

³To avoid any ambiguities with their Virasoro relatives, all quantities which refer to the superconformal algebra are distinguished by a hat. Note that \hat{c} should not be confused with $3c/2$ used in the ‘older literature’.

values of \hat{h} and \hat{c} by $|r, s\rangle$, suppressing the dependence on t . A Neveu-Schwarz (Ramond) Verma module $\hat{M}_{\hat{c}, \hat{h}}$ has a highest weight at level $rs/2$ if $\hat{c} = \hat{c}(t)$ and $\hat{h} = \hat{h}_{r,s}(t)$ for r, s positive integers with $r - s$ even (odd). For any pair of positive integers we shall define the space $\hat{\mathcal{F}}_{\hat{c}(t), \hat{h}_{r,s}(t)}$ as the quotient of $\hat{M}_{\hat{c}(t), \hat{h}_{r,s}(t)}$ by the Verma module $\hat{M}_{\hat{c}(t), \hat{h}_{r,s}(t) + rs/2}$ generated by the singular vector at level $rs/2$. We shall usually denote $\hat{\mathcal{F}}_{\hat{c}(t), \hat{h}_{r,s}(t)}$ by $\hat{\mathcal{F}}_{r,s}$ and $\hat{\mathcal{F}}_{1,1}$ by $\hat{\mathcal{F}}$. These spaces play an important role in Toda theory as the spaces of fields.

We shall take our conserved quantities to be polynomials in \hat{L} and \hat{G} , in which case they correspond to vectors in $\hat{\mathcal{F}}$. The possible embedding patterns of singular vectors are identical to those of the Virasoro algebra, and it is only in cases III₋ and III₊ (in the notation of [12]) that there are singular vectors in $\hat{\mathcal{F}}$ which may provide singular conserved quantities. From now on, we shall restrict to $t > 0$, in which case the values of \hat{c} for which $\hat{\mathcal{F}}$ has singular vectors are exactly the superconformal minimal models (including the limiting case $\hat{c} = 3/2$).

The superconformal minimal models are characterised by two positive integers (p, p') such that⁴

$$\frac{p - p'}{2} \text{ and } p' \text{ are relatively prime integers,} \quad (2.2)$$

with \hat{c} given by (2.1) with $t = p'/p$. The minimal model primary fields $\hat{\phi}_{r,s}$ have conformal dimensions $\hat{h}_{r,s}(t)$ where $1 \leq r < p'$, $1 \leq s < p$. The Verma module associated to $\hat{\phi}_{r,s}$ has primary singular vectors at level $rs/2$ and $(p' - r)(p - s)/2$ and an infinite number of secondary singular vectors. Fields with $r + s$ even are in the Neveu-Schwarz (NS) sector, while $r + s$ odd are in the Ramond (R) sector.⁵ Correspondingly, $\hat{\mathcal{F}}_{r,s}$ has a single singular vector of weight $\hat{h}_{r,s} + (p' - r)(p - s)/2 = \hat{h}_{r, 2p - s} = \hat{h}_{2p' - r, s}$, and in particular, $\hat{\mathcal{F}} = \hat{\mathcal{F}}_{1,1}$ has a unique singular vector at level $(p' - 1)(p - 1)/2 = \hat{h}_{1, 2p - 1} = \hat{h}_{2p' - 1, 1}$.

3 The singular-vector analysis of the perturbed N = 1 superconformal models

For a superconformal model, the perturbation may be either a NS or a R field. In the first case, the perturbation may either break or preserve supersymmetry. To cover all three possibilities we have to allow perturbations of both forms

$$\oint \frac{dz}{2\pi i} P(z), \quad \text{and} \quad \oint \frac{dz}{2\pi i} \hat{G}_{-1/2} P(z). \quad (3.1)$$

⁴ The fact that p, p' must satisfy eq. (2.2) can be most neatly seen from the non-unitary coset description [13]:

$$\frac{\widehat{su}(2)_m \oplus \widehat{su}(2)_2}{\widehat{su}(2)_{m+2}}, \quad \text{with} \quad m = \frac{s}{u} \quad (s, u) = 1, \quad m + 2 = \frac{2p'}{p - p'}, \quad p - p' = 2u.$$

The basic requirement is to have s and u coprime integers (and $u \geq 1$), which leads simply to this result.

⁵ Fields in the NS sector may be considered as superfields, that is, composed of two Virasoro primary fields, with conformal dimension differing by $1/2$. In a superspace formalism, a field ϕ in the NS sector can be decomposed as $\phi = \varphi + \theta\psi$; φ and ψ are usually referred to as the lower (dimension h) and the upper (dimension $h + 1/2$) components of the superfield respectively.

The first possibility covers the R and the supersymmetry-breaking NS perturbations, and the second the NS supersymmetry-preserving perturbations. Equally well, we shall consider conserved densities (which are now always NS fields – being vacuum descendants) of the two forms

$$\Phi_I, \quad \text{and} \quad \hat{G}_{-1/2}\Phi_I, \quad (3.2)$$

although we shall usually phrase our arguments in terms of the corresponding states

$$\chi_I = \Phi_I(0)|0\rangle, \quad \hat{G}_{-1/2}\chi_I = \hat{G}_{-1/2}\Phi_I(0)|0\rangle, \quad (3.3)$$

where χ_I is a NS highest-weight state. Finally, we need to consider the operator product of the perturbing field with the highest-weight state corresponding to the conserved density, and the leading term in the OPE may again be a highest-weight state χ_P or a descendant $\hat{G}_{-1/2}\chi_P$.

The simplest way for the density to be conserved is that the leading term in the OPE is a second-order pole.⁶ There are then three cases to consider, corresponding to whether the leading term is χ_P for P a NS field, χ_P for P a R field, or $\hat{G}_{-1/2}\chi_P$ for P a NS field. In the first case, the only possible field which can appear as the residue of the first order pole is the total derivative $\hat{L}_{-1}\chi_P$. In the second case, there may additionally be a term $\hat{G}_{-1}\chi_P$ and we must check that this does not appear. Finally, in the third case, the two fields which may appear are $\hat{L}_{-1}\hat{G}_{-1/2}\chi_P$ and $\hat{G}_{-3/2}\chi_P$ and we must again check the the second field does not appear.

Since we take $t = p'/p$ with p, p' satisfying (2.2), the only singular vector in $\hat{\mathcal{F}}_{r,s}$ is of weight $\hat{h}_{r,2p-s}$ and consequently, with $P = \hat{\phi}_{r,s}$, we identify $\chi_P = |r, 2p-s\rangle$, and $\chi_I = |1, 2p-1\rangle$, and the requirement that the leading pole be second order equates to a condition on the values of r, s, p and p' . To find this condition we first define

$$\Delta\hat{h} \equiv \hat{h}_P + \hat{h}_{\chi_I} - \hat{h}_{\chi_P} = \frac{1}{2} [(p' - r)(s - 1) + (p - s)(r - 1) + (r - 1)(s - 1)] . \quad (3.4)$$

As in the Virasoro case, the simplicity of the singular-vector argument depends crucially upon the fact that the above expression can be written as a sum of positive terms. We give in table 1 the eight possible operator product expansions, the value $\Delta\hat{h}$ must take for the leading pole to be of order two, whether P may be NS or R (indicated by a \checkmark) and whether the density is guaranteed to be conserved (indicated by $\checkmark\checkmark$)

We now analyse each of the possible values of $\Delta\hat{h}$ in turn.

3.1 $2\Delta\hat{h} = 2$

We are required to analyse

$$(p' - r)(s - 1) + (p - s)(r - 1) + (r - 1)(s - 1) = 2 . \quad (3.5)$$

⁶ As in the Virasoro case, no generic solutions are expected when the pole is not of second order.

Pert.	\times	Dens.	\rightarrow	$2\Delta\hat{h}$	NS	R	
$\hat{G}_{-1/2}P$	\times	$\hat{G}_{-1/2}\chi_I$	\rightarrow	χ_P	2	$\checkmark\checkmark$	
$\hat{G}_{-1/2}P$	\times	$\hat{G}_{-1/2}\chi_I$	\rightarrow	$\hat{G}_{-1/2}\chi_P$	3	\checkmark	
$\hat{G}_{-1/2}P$	\times	χ_I	\rightarrow	χ_P	3	$\checkmark\checkmark$	
$\hat{G}_{-1/2}P$	\times	χ_I	\rightarrow	$\hat{G}_{-1/2}\chi_P$	4	\checkmark	
P	\times	$\hat{G}_{-1/2}\chi_I$	\rightarrow	χ_P	3	$\checkmark\checkmark$	\checkmark
P	\times	$\hat{G}_{-1/2}\chi_I$	\rightarrow	$\hat{G}_{-1/2}\chi_P$	4	\checkmark	
P	\times	χ_I	\rightarrow	χ_P	4	$\checkmark\checkmark$	\checkmark
P	\times	χ_I	\rightarrow	$\hat{G}_{-1/2}\chi_P$	5	\checkmark	

Table 1: Possible operator products

To illustrate the procedure, we give all possible results in table 2 with r, s, p, p' positive integers, and indicate whether or not this corresponds to a minimal model representation P .

$(p-s)(r-1) + (p'-r)(s-1) + (r-1)(s-1) = 2$	(r, s)	(p, p')	NS	R	Min.
2		$\begin{cases} (2, 1) & (3, -) \\ (3, 1) & (2, -) \end{cases}$	\checkmark	\checkmark	\checkmark
1		No solutions			
1		$(2, 2) \quad (3, 2)$			
0		$\begin{cases} (1, 2) & (-, 3) \\ (1, 3) & (-, 2) \end{cases}$	\checkmark	\checkmark	\checkmark
0		$(2, 2) \quad (2, 3)$			
0		$\begin{cases} (2, 3) & (3, 2) \\ (3, 2) & (2, 3) \end{cases}$		\checkmark	

Table 2: Solutions to $\Delta\hat{h} = 1$

As a result, we have four solutions to the equation $2\Delta\hat{h} = 2$ which have minimal values of p and p' . Of these, two have NS perturbations and two have R perturbations. However, if we look back at table 1, we see that only the NS perturbations ever have $2\Delta\hat{h} = 2$, and so the R solutions are spurious. Moreover, the remaining two NS solutions are related by interchange of p and p' ; requiring $p' < p$ leaves us with a single solution.

Furthermore, the conserved densities are guaranteed to be conserved in this case. We have thus found our first integrable perturbation of a superconformal theory by this method, which is

$$\hat{G}_{-1/2}P = \hat{G}_{-1/2}\hat{\phi}_{1,3}, \quad \hat{G}_{-1/2}\chi_I = \hat{G}_{-1/2}\hat{\phi}_{3,1}, \quad \chi_P = \hat{\phi}_{3,3}, \quad , \quad (3.6)$$

where $p' = 2$ and consequently $p = 4k$. In these cases, the weight of the conserved density χ_I is $\hat{h}_{3,1}(2/4k) + \frac{1}{2} = 2k$ (cf. [6]).

The detailed analysis of the next three cases – $\Delta\hat{h} = 3/2, 2$ and $5/2$ – is reported in appendix D (where the results are presented in the form of tables). In the main text, we confine ourself to the interpretation of the possible solutions.

3.2 $2\Delta\hat{h} = 3$

In this case we find no NS perturbations in minimal models, and instead two Ramond perturbations,

$$P = \hat{\phi}_{1,2}, \quad \hat{G}_{-1/2}\chi_I = \hat{G}_{-1/2}\hat{\phi}_{7,1}, \quad p' = 4, \quad p = 4k + 2, \quad (3.7)$$

$$P = \hat{\phi}_{1,4}, \quad \hat{G}_{-1/2}\chi_I = \hat{G}_{-1/2}\hat{\phi}_{3,1}, \quad p' = 2, \quad p = 4k. \quad (3.8)$$

However, using the values of the coefficients of the terms $\hat{G}_{-1}\chi_P$ which appear in the residue of the operator product expansion, and which are given in eqs C.8, we see that these do not give conserved quantities.

3.3 $2\Delta\hat{h} = 4$

In this case we find both NS and R perturbations in minimal models. In the Neveu-Schwarz case, there are six possibilities:

$$\begin{aligned} a) \quad & \hat{G}_{-1/2}P = \hat{G}_{-1/2}\hat{\phi}_{1,3}, & \chi_I &= \hat{\phi}_{5,1}, & p' &= 3, & p &= 2k + 3, \\ b) \quad & \hat{G}_{-1/2}P = \hat{G}_{-1/2}\hat{\phi}_{1,5}, & \chi_I &= \hat{\phi}_{3,1}, & p' &= 2, & p &= 4k, \\ c) \quad & P = \hat{\phi}_{1,3}, & \hat{G}_{-1/2}\chi_I &= \hat{G}_{-1/2}\hat{\phi}_{5,1}, & p' &= 3, & p &= 2k + 3, \\ d) \quad & P = \hat{\phi}_{1,5}, & \hat{G}_{-1/2}\chi_I &= \hat{G}_{-1/2}\hat{\phi}_{3,1}, & p' &= 2, & p &= 4k, \\ e) \quad & P = \hat{\phi}_{1,3}, & \chi_I &= \hat{\phi}_{5,1}, & p' &= 3, & p &= 2k + 3, \\ f) \quad & P = \hat{\phi}_{1,5}, & \chi_I &= \hat{\phi}_{3,1}, & p' &= 2, & p &= 4k, \end{aligned}$$

and only one in the Ramond case,

$$g) \quad P = \hat{\phi}_{1,2}, \quad \chi_I = \hat{\phi}_{9,1}, \quad p' = 5, \quad p = 2k + 5.$$

For the Ramond case, the calculation (C.7) of the coefficient of $\hat{G}_{-1}\chi_P$ in the residue of the OPE shows that the density is not conserved.

For cases a), b), c) and d), the calculation (B.6) of the coefficient of $\hat{G}_{-3/2}\chi_P$ in the residue of the OPE shows that the density is not conserved.

In cases e) and f) the densities are guaranteed to be conserved.

3.3.1 Case (e)

Here we have found that the perturbation by $\oint \hat{\phi}_{1,3}$ has singular conserved densities for $\hat{c}(3/(2k+3))$. These are of weight $2k+2$, so that we expect that there are conserved quantities for the $\hat{\phi}_{1,3}$ perturbation of weights $2, 4, 6, 8, \dots$ which become singular for these \hat{c} -values.

3.3.2 Case (f)

Here we have found that the perturbation by $\oint \hat{\phi}_{1,5}$ has singular conserved densities for $\hat{c}(2/4k)$. These are of weight $2k - 1/2$, so that we expect that there are conserved quantities

for the $\hat{\phi}_{1,5}$ perturbation of weights $3/2, 7/2, 11/2, 15/2, \dots$ which become singular for these \hat{c} -values.

3.4 $2\Delta\hat{h} = 5$

In that case, we have

$$(p' - r)(s - 1) + (p - s)(r - 1) + (r - 1)(s - 1) = 5 \quad (3.9)$$

Taking $p' < p$, there are three solutions: $(r, s) = (2, 2)$, $p' = 3$ and $p = 5$, $(r, s) = (1, 2)$ with $p' = 6$ and $(r, s) = (1, 6)$ with $p' = 2$. For the first case, the perturbing field $\hat{\phi}_{2,2}$ is identical to $\hat{\phi}_{1,3}$; this is a solution already found previously, but viewed here from a different form of the quotient space \mathcal{F} (i.e., the remaining primitive singular vector differs in the two cases). The other two correspond to a Ramond perturbing field, which is incompatible with $2\Delta\hat{h} = 5$ (cf. table 1.)

3.5 Summary

We have found that the following perturbations have singular conserved densities at the indicated values of c :

- 1 The supersymmetry-preserving perturbation $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ with singular conserved densities of weight $2k$ for $c = \hat{c}(2/4k)$.
- 2 The supersymmetry-breaking perturbation $\hat{\phi}_{1,3}$ with singular conserved densities of weight $2k + 2$ for $c = \hat{c}(3/(2k + 3))$.
- 3 The supersymmetry-breaking perturbation $\hat{\phi}_{1,5}$ with singular conserved densities of weight $2k - 1/2$ for $c = \hat{c}(2/4k)$.

4 Direct construction of the conservation laws

We can now investigate by explicit computation whether the singular conserved densities we have predicted exist, and whether they also exist for generic c -values. We give the results for the perturbations $\hat{G}_{-1/2}\hat{\phi}_{1,3}$, $\hat{\phi}_{1,3}$ and $\hat{\phi}_{1,5}$ in table 3.

In each case we have considered the equations for a general field in $\hat{\mathcal{F}}$ of low level to be a conserved density. In the first column we give the values of $t = p'/p$ for which there is a conserved quantity, and the multiplicity of these quantities in parentheses if it is more than one, and ‘all’ if there is a conserved quantity at that level for all t (and hence for all c). (For instance, for the $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ perturbation we see that there are two conserved densities with $\Delta = 4$ when $t = 1$; since there is a generic conserved density at that level (indicated by the ‘all’), there is thus one extra conserved density at $t = 1$.)

In each case, we have checked whether the conserved density is a highest weight, whether it is the super-descendant $\hat{G}_{-1/2}\psi$ of some field ψ (modulo total derivatives) and whether that state ψ is itself ever a highest-weight state in $\hat{\mathcal{F}}$. The number of densities of each form is again given in parentheses. (Again considering the $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ perturbation, we see that the generic conserved density of weight 4 is always a super-descendant $\hat{G}_{-1/2}\psi$ and that ψ is singular for $t = 1/4, 4$. In the special case $t = 1$, only one of the two conserved densities of weight 4 is a super-descendant and it is not singular.)

As can be seen from the tables, in each case the singular conserved quantities that we have predicted do exist and that they are indeed highest weights (respectively super-partners of highest weights) for supersymmetry-breaking (resp. supersymmetry-preserving) perturbations. Moreover, as for all the cases considered in [1], for the perturbations $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ and $\hat{\phi}_{1,3}$, these conserved quantities also exist for **generic** values of c .

However, a new phenomenon is observed for the case of the $\hat{\phi}_{1,5}$ perturbation, which is that the predicted fermionic singular conserved quantities do not generalise to **generic** values of c . Secondly, we find that there are in fact conserved densities for generic c -values of all even weights, and on closer inspection, these are identical to the conserved densities of the $\hat{\phi}_{1,3}$ perturbation modulo p, p' interchange.

We now make some comments on the results of the singular-vector analysis and the results in table 3.

(i): For each perturbation there are some values of t at which there are extra conservation laws over and above those present for generic t , namely at $t = 1, 2$ for $\hat{G}_{-1/2}\hat{\phi}_{1,3}$, at $t = 1, 3/2$ for $\hat{\phi}_{1,3}$ and at $t = 2/3, 1/(2k)$ for $\hat{\phi}_{1,5}$. Each of these cases can be understood from the free-field construction in sections 6-7 and in certain cases from the presence of identifiable extra symmetries, which we detail in section 8.

(ii): As mentioned in the introduction, on the basis of the free-field construction and the connection to affine Toda field theory, we expect the conservation laws to satisfy a duality relation, namely that the conserved quantities of the perturbation $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ be invariant under $t \rightarrow 1/t$, and that those of $\hat{\phi}_{1,3}$ and $\hat{\phi}_{1,5}$ be swapped under $t \rightarrow 1/t$. This is indeed observed for generic values of t . However, we notice some surprising failures of duality in the singular-vector analysis and in table 3, namely:

(ii)(a): The conservation laws of weight $2k - 1/2$ of $\hat{\phi}_{1,5}$ at $t = 1/(2k)$ are not present for $\hat{\phi}_{1,3}$ at $t = 2k$.

(ii)(b): The extra conservation laws at $t = 1$ for $\hat{\phi}_{1,3}$ are not present for $\hat{\phi}_{1,5}$.

(ii)(c): The extra conservation laws at $t = 2$ for $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ are not present at the dual value of t , $1/2$.

In each case these are ‘enhancements’ of the spectrum of conserved quantities as mentioned in (i), which are not shared by the dual model, as we explain in section 6.

(iii): For all perturbations, the higher conservation laws for **generic** t are bosonic. However, for special t -values there are also fermionic conservation laws, even for the supersymmetry-

Δ	$\hat{G}_{-1/2} \hat{\phi}_{1,3}$				$\hat{\phi}_{1,3}$				$\hat{\phi}_{1,5}$			
	t	hwt	$\hat{G}_{-1/2} \psi$	$\hat{G}_{-1/2}$ hwt	t	hwt	$\hat{G}_{-1/2} \psi$	$\hat{G}_{-1/2}$ hwt	t	hwt	$\hat{G}_{-1/2} \psi$	$\hat{G}_{-1/2}$ hwt
3/2	all (1)	$t = 2, 1/2$	—	—	—				—			
2	all (1)	—	✓	$t = 1/2, 2$	all (1)	—	✓	$t = 1/2, 2$	all (1)	—	✓	$t = 1/2, 2$
5/2	—				—				—			
3	—				—				—			
7/2	1 (1)	—	—		—				1/4 (1)	✓	—	
	2 (1)	—	—									
4	all (1)	—	✓	$t = 1/4, 4$	all (1)	$t = 3/5$	$t = 2$	—	all (1)	$t = 5/3$	$t = 1/2$	—
	1 (2)	—	(1)	—								
9/2	1 (1)	—	(1)	—	1 (1)	—	(1)	—	—			
5	—				—				—			
11/2	1 (2)	—	—		—				1/6 (1)	✓	—	
	2 (1)	—	—									
6	all (1)	—	✓	$t = 1/6, 6$	all (1)	$t = 3/7$	$t = 2$	—	all (1)	$t = 7/3$	$t = 1/2$	—
	1 (4)	—	(2)	—	3/2 (2)	—	—		2/3 (2)	—	—	
	2 (2)	—	(1)	—								
13/2	1 (2)	—	(2)	—	1 (1)	—	(1)	—	—			
	2 (1)	—	(1)	—								
7	1 (1)	—	—		—				—			
15/2	1 (5)	—	(1)	—	1 (1)	—	—		1/8 (1)	✓	—	
	2 (2)	—	—									
8	all (1)	—	✓	$t = 1/8, 8$	all (1)	—	$t = 2$	—	all (1)	—	$t = 1/2$	—
	1 (7)		(4)	—	1 (2)	—	(1)	—				
	2 (3)	—	(2)	—	3/2 (3)	—	—		2/3 (3)	—	—	
17/2	1 (4)	—	(3)	—	1 (1)	—	(1)	—	—			
	2 (1)	—	(1)	—								
9	1 (4)	—	(1)	—	3/2 (1)	—	—		2/3 (1)	—	—	
	2 (1)	—	—									

Table 3: The conserved densities of the integrable perturbations $\hat{G}_{-1/2} \hat{\phi}_{1,3}$, $\hat{\phi}_{1,3}$ and $\hat{\phi}_{1,5}$

breaking perturbations. In that respect, the following two observations are worth making.

(iii)(a): As already indicated in the introduction (cf. the analysis of appendix A), fermionic conservation laws are not a sure sign of integrability, even if there are an infinite number of them.

(iii)(b): The fermionic conservation laws do not necessarily anticommute amongst themselves. Actually, the Jacobi identity forces this anticommutator to either vanish or be equal to a bosonic conservation law. We have checked that the higher ($\Delta \geq 7/2$) fermionic singular conservation laws of the $\hat{\phi}_{1,5}$ perturbation square to give the bosonic conservation laws of appropriate dimension – and, obviously, the latter vanishes when all singular vectors are set to zero.⁷ Hence, whereas for the $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ and $\hat{\phi}_{1,3}$ perturbations, every generic conservation law becomes singular at a particular value of c , in the $\hat{\phi}_{1,5}$ case, a generic conservation law never becomes singular but its square root does at a particular value of c !

5 Affine Toda theory, Lie superalgebras and duality

The rest of the article is devoted to the analysis of the data presented in the previous section. This analysis will rely heavily on the free-field construction. Our first aim is to unravel the origin of the generic duality which is a symmetry of the supersymmetry-preserving perturbation and relates the two supersymmetry-breaking ones. This is done in two steps. In the present section, we introduce affine Toda field theories (defined in terms of Lie superalgebras) and show that the three integrable perturbations can be put in a one-to-one correspondence with the three affine Lie superalgebras having exactly two simple roots. Duality, together with its occasional breakdown, is then analysed in the following section.

The super Liouville field theory of one bosonic and one fermionic field, with Lagrangian density

$$\frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + i \bar{\psi} \not{\partial} \psi \right) - 2i\alpha \bar{\psi} \psi e^{-i\phi/\alpha} , \quad (5.1)$$

provides a model for the superconformal minimal models. In particular it has $N = 1$ superconformal symmetry with central charge $c = \hat{c}(t)$ as in (2.1) with $t = 1/\alpha^2$, and the fields

$$\hat{\phi}_{1,m} = \exp \left(i \frac{(m-1)}{2\alpha} \phi \right) \quad (5.2)$$

transform as the appropriate Neveu-Schwarz primary fields of weight $\hat{h}_{1,m}(t)$.

Olshanetsky was the first to realise that one can define integrable field theories associated to Lie superalgebras, generalising the bosonic affine Toda field theories to theories containing fermions [15]. In particular, there are three integrable field theories associated to Lie superalgebras which describe one bosonic and one fermionic field, and which can be identified as perturbations of the super-Liouville theory. They are associated to the algebras $B^{(1)}(0,1)$, $C^{(2)}(2)$ and $A^{(4)}(0,2)$, whose Dynkin diagrams are given in table 1.

⁷This is reminiscent of the algebra of the fermionic nonlocal charges for the classical supersymmetric Korteweg-de Vries equation [14]: a fermionic charge exists at each half-integer degree and the Poisson bracket of any such charge with itself is equal to a bosonic local charge.

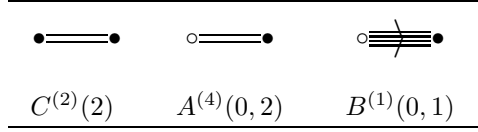


Figure 1: The Dynkin diagrams of $C^{(2)}(2)$, $A^{(4)}(0,2)$ and $B^{(1)}(0,1)$

As explained by Liao and Mansfield [16], one can simply set the auxiliary fields to zero in the quantum theory. The argument goes as follows. The auxiliary fields are necessary for the classical on-shell integrability. Since they appear without derivatives, they can always be eliminated. Classically, the auxiliary fields are found (by solving the equations of motion) to be products of exponentials. It turns out that in the quantum theory, at imaginary coupling, these products of exponentials vanish identically due to normal ordering effects. Hence, they can be ignored. The remaining terms in the Lagrangians are the standard kinetic terms

$$\frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + i \bar{\psi} \not{\partial} \psi \right) , \quad (5.3)$$

and the potentials:

$$\begin{array}{ccc} -2i\alpha \bar{\psi} \psi (e^{-i\phi/\alpha} + e^{i\phi/\alpha}) & -2i\alpha \bar{\psi} \psi e^{-i\phi/\alpha} - \alpha^2 e^{i\phi/\alpha} & -2i\alpha \bar{\psi} \psi e^{-i\phi/\alpha} - \alpha^2 e^{2i\phi/\alpha} \\ C^{(2)}(2) & A^{(4)}(0,2) & B^{(1)}(0,1) \end{array} \quad (5.4)$$

As can be seen directly (i.e., by reading off the perturbation from the second part of the potential), these three theories correspond to the three integrable perturbations $\hat{G}_{-1/2}\hat{\phi}_{1,3}$, $\hat{\phi}_{1,3}$ and $\hat{\phi}_{1,5}$ that we have found.

The conserved quantities that we are interested in are polynomials in $\partial\phi$ and ψ which commute with the terms in the potentials (5.4), whose integrals are combinations of terms of the form:

$$Q^B(\beta) = \oint \frac{dz}{2\pi i} : \exp(i\beta\sqrt{2}\phi) : \quad \text{and} \quad Q^F(\beta) = \oint \frac{dz}{2\pi i} : \psi \exp(i\beta\phi) : , \quad (5.5)$$

for various values of β . The spaces of fields polynomial in $\partial\phi$ and ψ which commute with these operators have been studied extensively since $Q^B(\beta)$ and $Q^F(\beta)$ arise as screening charges for the free-field constructions of the Virasoro and $N = 1$ super-Virasoro algebras respectively. We shall review in some detail the results for the bosonic screening charge Q^B and then present the results for Q^F much more briefly; after this digression, we will discuss the implications of this analysis for the conserved quantities of the affine Toda theories.

6 The free-field construction of the Virasoro algebra and the screening charge

Before plunging into this somewhat technical section, let us reformulate our motivations. In the previous section, we have introduced a correspondence between the integrable perturba-

tions and Toda theories. In this section, we study the conservation laws of the latter and their relations in theories obtained by duality with the aim of transposing these results in the context of perturbed conformal theories. This last point is addressed in the next section.

Conserved integrals in the (g, β) affine Toda theory commute with the defining potential (given in (6.8)) which is a sum of two charges Q . Hence these conservation laws lie in the intersections of the kernels of the two charges. The analysis of this section relies on this latter point of view and the emphasis is in the comparison of the kernel of a screening charge $Q^B(\beta)$ or $Q^F(\beta)$ with that of the dual theory obtained by replacing β with $-1/\beta$.

The presentation of this analysis is facilitated by considering first the Virasoro case, hence by concentrating on the kernel of $Q^B(\beta)$. With minor modifications, these results will be used for the description of the bosonic part of the $A^{(4)}(0, 2)$ and $B^{(1)}(0, 1)$ potentials.

The operator $Q^B(\beta)$ has been much studied in conformal theory because it commutes with the generator of the Virasoro algebra

$$L_\beta = -\frac{1}{2} : (\partial\phi)^2 : + \frac{i}{\sqrt{2}} (\beta - \frac{1}{\beta}) \partial^2 \phi , \quad (6.1)$$

a property which makes it a key ingredient in the calculation of correlation functions [17]. A more precise formulation of this computational approach [18] led to a characterisation of the irreducible highest-weight representations of the Virasoro algebra as cohomology spaces using maps between Fock spaces, maps which can be considered as products of $Q^B(\beta)$. For this reason, the kernel of $Q^B(\beta)$ has been extensively studied.

In the following three subsections, we describe the Virasoro structure of the Fock spaces, the relevant maps relating the different Fock spaces and finally the relationship between the kernels of $Q^B(\beta)$ and $Q^B(-1/\beta)$. These results are extended to the supersymmetric case in the remaining subsections.

6.1 The free-field Fock space representations

The modes of L_β (6.1) generate the Virasoro algebra of central charge $c = 13 - 6t - 6/t$ where $t = \beta^{-2}$. They act on Fock spaces F_μ on which $\alpha_0 = \oint i\partial\phi dz/(2\pi i)$ takes the value μ . F_μ is a highest-weight representation of the Virasoro algebra with highest weight $|\mu\rangle$ whose conformal weight is

$$h(\mu) = \frac{1}{2}\mu^2 - \frac{\mu}{\sqrt{2}}(\beta - 1/\beta) . \quad (6.2)$$

The structure of F_μ as a representation of the Virasoro algebra for all μ, β is given by Feigin and Fuchs in [12]. For our purposes it is only necessary to consider F_0 and β real. We can thus limit our attention to their sub-cases $\text{II}_-, \text{II}_+(+), \text{III}_-^{00}, \text{III}_-^0(\pm)$ and III_- , since F_0 with β real must fall into one of these classes.

The classification of F_μ is given by the number of solutions of the equation

$$\mu = \mu_{r,s} \equiv \frac{(1-r)\beta}{\sqrt{2}} - \frac{(1-s)}{\beta\sqrt{2}} , \quad (6.3)$$

with integral values of r and s . Those cases of interest to us for which (6.3) gives a single solution are as follows:

II_- : a single solution with $rs < 0$.

$\text{II}_+(+)$: a single solution with $r > 0, s > 0$.

For the cases III_-^* we must have $t = p'/p$ with p', p coprime positive integers, and μ must be expressible as $\mu_{m+jp', m'}$ with integers j, m, m' , $0 \leq m < p', 0 \leq m' < p$. Note that exactly in these cases μ does not have a unique representation in the form (6.3) but satisfies $\mu_{r,s} = \mu_{r+kp', s+kp}$ for any k .

III_-^{00} : $\mu = \mu_{jp', 0}$.

$\text{III}_-^0(+)$: (i) $\mu = \mu_{m+jp', 0}$ or (ii) $\mu = \mu_{0, m+jp}$ with $j < 0$.

$\text{III}_-^0(-)$: (i) $\mu = \mu_{m+jp', 0}$ or (ii) $\mu = \mu_{0, m+jp}$ with $j \geq 0$.

III_- : $\mu = \mu_{m+jp', m'}$ where $0 < m < p'$ and $0 < m' < p$.

The description of the representations F_μ in [12] is in terms of embedding diagrams; the cases of interest are presented in table 2. In these tables, the vertices \bullet correspond to highest weights of the Virasoro algebra; the vertices \circ correspond to highest weights in the quotient of F_μ by the module generated by the vertices \bullet ; finally, the vertices \square correspond to highest weights in the quotient of this latter module by the submodule generated by the vertices \circ . The special vertex \odot corresponds to a vector of type \circ which is also a highest-weight vector itself. Finally, an arrow pointing from a to b indicates that b is in the submodule generated by a .

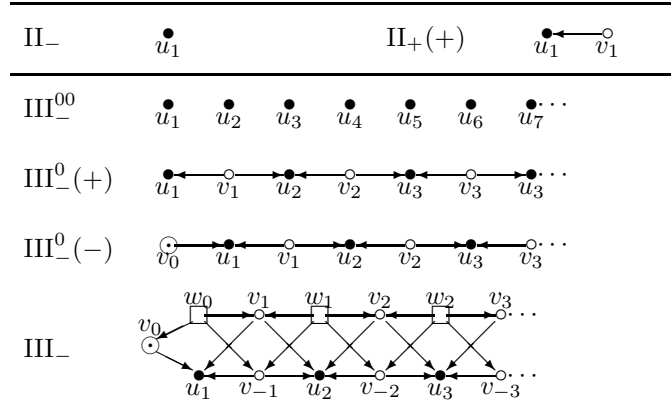


Figure 2: The embedding diagrams of types II_- , $\text{II}_+(+)$ and III_-^*

The dimensions of the vertices, taken from [19] (with $p \leftrightarrow p'$), are:

$$\begin{aligned}
\text{II}_- & h(u_1) = h_{r,s} \\
\text{II}_+(+) & h(u_1) = h_{r,s} & h(v_1) = h_{-r,s} \\
\text{III}_-^{00} & h(u_i) = h_{(|j|+2i)p',0} \\
\text{III}_-^0(+) & h(u_i) = \begin{cases} h_{m+(j-2i+2)p',0} \\ h_{0,m'+(j-2i+2)p} \end{cases} & h(v_i) = \begin{cases} h_{m-(j-2i+2)p',0} \\ h_{0,m'-(j-2i+2)p} \end{cases} \\
\text{III}_-^0(-) & h(u_i) = \begin{cases} h_{m-(j+2i)p',0} \\ h_{0,m'-(j+2i)p} \end{cases} & h(v_i) = \begin{cases} h_{m+(j+2i)p',0} \\ h_{0,m'+(j+2i)p} \end{cases} \\
\text{III}_- & h(u_i) = h_{-m+(|j|+2i)p',m'} & h(v_i) = \begin{cases} h_{m+(|j|+2i)p',m'} & i > 0 \\ h_{m-(|j|-2i)p',m'} & i < 0 \end{cases} \\
& h(w_i) = h_{-m-(|j|+2i)p',m'}
\end{aligned} \tag{6.4}$$

6.2 Maps between the Fock spaces

As already indicated, $Q^B(\beta) = : \exp(i\beta\sqrt{2}\phi) :$, being a primary field of weight 1 with respect to the Virasoro algebra (6.1), commutes with the modes of $L_\beta(z)$. As a result it is possible to define different maps $Q_+^{(k)}$ and $\widehat{Q}_+^{(k)}$ between the various spaces F_μ from appropriate k -folded products of $Q^B(\beta)$. (Such products require a prescription for the multiple contour integrations and this rule distinguishes $Q_+^{(k)}$ from $\widehat{Q}_+^{(k)}$; the one used in the definition of $\widehat{Q}_+^{(k)}$ insures that it acts nontrivially between some spaces of interest at the boundary of the Kac table – see [19] for details). These operators have the following properties:

- a) $Q_+^{(1)} = \widehat{Q}_+^{(1)} = Q^B(\beta)$. We shall also denote this by Q_+ for short.
- b) $Q_+^{(r)} Q_+^{(s)} = Q_+^{(r+s)}$, when $0 \leq r, s, r+s \leq p'$.
- c) $Q_+^{(p')} = 0$ for $p' > 1$.
- d) $\widehat{Q}_+^{(p')} \neq 0$ for $p' \geq 1$.

From Theorem 3.1, Lemmas 4.1 and 5.1 of [19], the action of the various maps $Q_+^{(m)}$, $Q_+^{(p'-m)}$ and $\widehat{Q}_+^{(p')}$ between the spaces defined above are found to be as in figure 3.

It is now straightforward to read off the kernel of Q_+ on $F_{\mu_{1,1}}$ for each value of $t > 0$.

t irrational: $F_{\mu_{1,1}}$ is of type $\text{II}_+(+)$ and if we consider Fig. 3(a) with $r = s = 1$, we find that the kernel of $Q_+ = Q_+^{(r)}$ on F_0 is given by the Virasoro submodule generated by the vector u_1 .

$t = 1$: In this case $F_{\mu_{1,1}} = F_{\mu_{0,0}}$ is of type III_-^{00} , and if we consider Fig. 3(b) with $p' = 1$, we find that the kernel of $Q_+ = \widehat{Q}_+^{(p')}$ on F_0 is given by the Virasoro submodule generated by the vector u_1 .

$t = p'$, p' integer: In this case $F_{\mu_{1,1}} = F_{\mu_{1-p',0}}$ is of type $\text{III}_-^0(+)$ and if we consider Fig. 3(c) with $m = p' - 1$, we find that the kernel of $Q_+ = Q_+^{(p'-m)}$ on F_0 is given by the Virasoro submodule generated by the vectors u_1, u_2, u_3, \dots

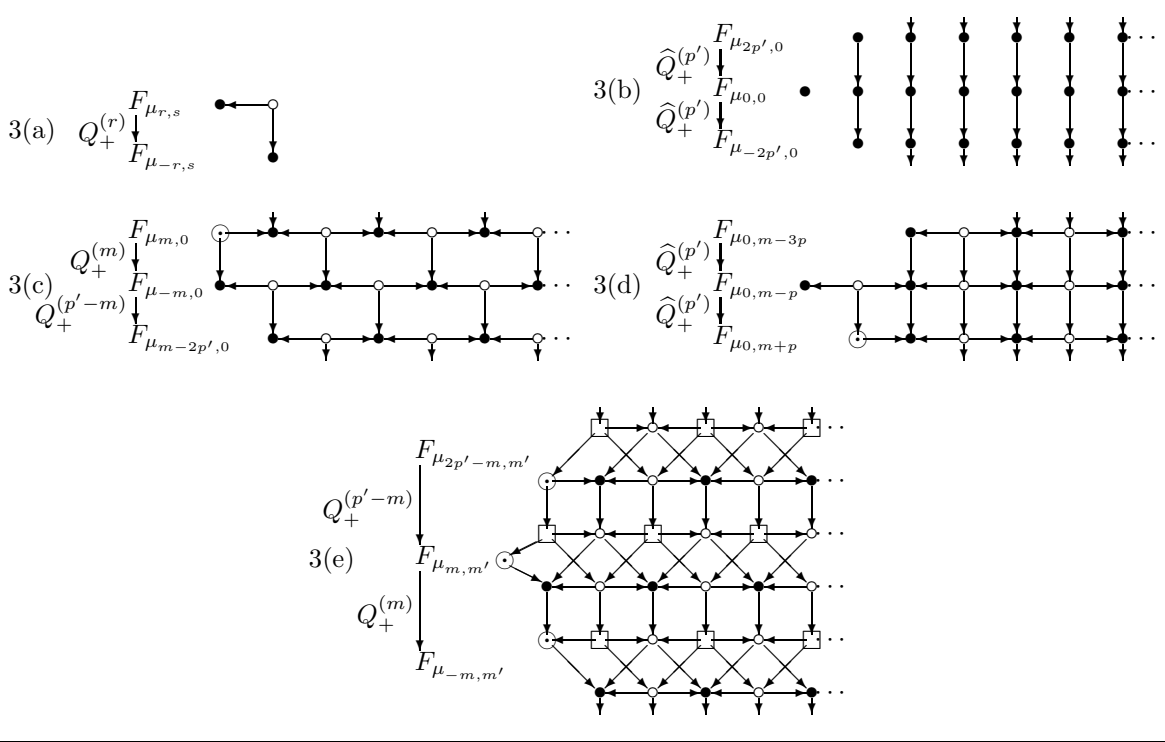


Figure 3: The maps between the Fock modules of types $\text{II}_-, \text{II}_+(+)$ and III_-^*

$t = 1/p$, p integer: In this case $F_{\mu_{1,1}} = F_{\mu_{0,1-p}}$ is of type $\text{III}_-^0(+)$ and if we consider Fig. 3(d) with $p' = 1$ we find that the kernel of $Q_+ = \widehat{Q}_+^{(p')}$ on F_0 is given by the Virasoro submodule generated by the vector u_1 .

$t = p'/p$, p', p integers greater than 1: In this case $F_{\mu_{1,1}}$ is of type III_- and if we consider Fig. 3(e) with $m = m' = 1$, we find that the kernel of $Q_+ = Q_+^{(m)}$ on F_0 is given by the Virasoro submodule generated by the vectors $v_0, w_0, v_1, w_1, v_2, w_2, \dots$

6.3 The kernels of $Q^B(\beta)$ and $Q^B(-1/\beta)$

The first point to make is that $L_\beta = L_{-1/\beta}$ and $\mu_{1,1} = 0$ is invariant under $\beta \rightarrow -1/\beta$, so that the kernels of $Q^B(\beta)$ and $Q^B(-1/\beta)$ are related.

It is clear that in the case $\beta^2 = 1/t$ irrational they are in fact equal, being exactly the Virasoro submodule of $F_{\mu_{1,1}}$ generated by the highest-weight state $|0\rangle$.

Similarly for $t = 1$, the kernels of $Q^B(1)$ and $Q^B(-1)$ are again equal, being again exactly the Virasoro submodule of $F_{\mu_{1,1}}$ generated by the highest-weight state $|0\rangle$. However for all other rational values of β^2 , one or other of the kernels is increased as follows:

For $t = 1/\beta^2 = 1/p$ for $p > 1$ integer, we have the kernel of $Q^B(\beta)$ again being simply the

submodule generated by $u_1 = |0\rangle$, whereas for the dual values $t = 1/\beta^2 = p' > 1$, the kernel of $Q^B(-1/\beta)$ is much larger being generated by u_1, u_2, \dots , so that

$$\ker Q^B(-1/\sqrt{k}) \supset \ker Q^B(\sqrt{k}), \quad k = 2, 3, \dots \quad (6.5)$$

Notice that the argument does not depend upon the sign of β which means that we also have

$$\ker Q^B(1/\sqrt{k}) \supset \ker Q^B(-\sqrt{k}), \quad k = 2, 3, \dots \quad (6.6)$$

For generic $t = 1/\beta^2 = p'/p$ with $p', p > 1$ coprime integers, the kernel of $Q^B(\beta)$ is generated by $v_0, w_0, v_1, w_1, \dots$. For the dual value of β the kernel of $Q^B(-1/\beta)$ is generated by a similar submodule of $F_{\mu_{1,1}}$, but with respect to the labelling in the first case it is generated by $v_0, w_0, v_{-1}, w_1, v_{-2}, w_2, \dots$. It is clear that the kernels are not identical but it is hard to tell whether one is systematically larger than the other.

6.4 The free-field construction of the $N = 1$ super-Virasoro algebra and the screening charge

The free-field construction of the $N = 1$ super-Virasoro algebra in terms of one free boson ϕ and one free fermion ψ has also been known for a long time [20]. The algebra generators take the form

$$\hat{L}_\beta(z) = -\frac{1}{2}[:(\partial\phi)^2: + :\psi\partial\psi:] + \frac{i}{2}(\beta - \frac{1}{\beta})\partial^2\phi, \quad \hat{G}_\beta = i\partial\phi\psi + (\beta - \frac{1}{\beta})\partial\psi, \quad (6.7)$$

and the screening charge is $Q^F(\beta)$ as in (5.5). The Neveu-Schwarz Fock spaces $\hat{F}_{\hat{\mu}_{r,s}}$ with $\hat{\mu}_{r,s} = (1/2)((1-r)\beta - (1-s)/\beta)$ and $r+s$ even are Neveu-Schwarz highest-weight representations of the $N = 1$ super-Virasoro algebra with $c = \hat{c}(t)$ and $h = \hat{h}_{r,s}(t)$ as in (2.1) with $t = 1/\beta^2$.

The analysis of the structure of these Fock representations as representations of the $N = 1$ super-Virasoro algebra carry through exactly as for the Virasoro algebra. The classification of Fock modules is still given by the number of solutions to $\mu = \hat{\mu}_{r,s}$, with the proviso that $r-s$ must be even. The analysis of the kernel of $Q^F(\beta)$ is also the same since the maps between the spaces are unchanged: one only needs to replace $h_{r,s}$ by $\hat{h}_{r,s}$, M by \hat{M} , Q^B by Q^F etc. wherever appropriate.

Accordingly, the classification of the vacuum Fock representation \hat{F}_0 is as follows:

t irrational means that \hat{F}_0 is of type $\text{II}_+(+)$.

$t = 1$ means that \hat{F}_0 is of type III_-^{00} .

$t = 1/(2k+1)$ or $t = (2k+1)$ for k a positive integer means that \hat{F}_0 is of type $\text{III}_-^0(+)$.

For all other rational values of t , including $t = 2k, 1/(2k)$, \hat{F}_0 is of the minimal type, i.e., III_- .

6.5 The kernels of $Q^F(\beta)$ and $Q^F(-1/\beta)$

Since $\hat{L}_\beta = \hat{L}_{-1/\beta}$ and $\hat{G}_\beta = \hat{G}_{-1/\beta}$, we know that the kernels of $Q^F(\beta)$ and $Q^F(-1/\beta)$ are related. As for the bosonic case, we can say that $\ker Q^F(\beta) = \ker Q^F(-1/\beta)$ if β^2 is irrational or takes the value 1. If $\beta^2 = 2k + 1$ then $\ker Q^F(\beta) \subset \ker Q^F(-1/\beta)$. In all other cases there is no clear relationship between the two kernels.

6.6 The conserved quantities of the affine Toda theories

The conserved quantities of the affine Toda theories are given as the intersections of the kernels of the potential terms. Ignoring the \bar{z} dependent terms and the constant prefactors of each term (which are irrelevant for the holomorphic conserved quantities), we write these in the following way:

$$\begin{array}{ccc} Q^F(\beta) + Q^F(-\beta) & Q^F(\beta) + Q^B(-\beta/\sqrt{2}) & Q^F(\beta) + Q^B(-\beta\sqrt{2}) \\ C^{(2)}(2) & A^{(4)}(0, 2) & B^{(1)}(0, 1) \end{array} \quad (6.8)$$

From the discussion in sections 6.3 and 6.4, it is clear that for β^2 positive and irrational, the conserved quantities of the theories $C^2(2)(\beta)$ and $C^2(2)(-1/\beta)$ are equal, as are those of $A^{(4)}(0, 2)(\beta)$ and $B^{(1)}(0, 1)(-1/\beta)$. Further evidence for these relations comes from the equivalence of the S-matrices of the corresponding theories for β imaginary.

For β^2 positive and rational, our analysis does not allow us to say whether the conserved densities of a theory are equal to those of its dual. However, there are special circumstances in which we can be certain that the number of conserved quantities of one theory are no smaller than the conserved quantities in the dual theory. But this is so only for $C^{(2)}(2)$ (whose potential is solely given in terms of Q^F), when $\beta^2 = 3, 5, \dots$. In that case, $\ker Q^F(\pm\beta) \subset \ker Q^F(\mp 1/\beta)$, so that the intersection satisfies

$$(\ker Q^F(\beta) \cap \ker Q^F(-\beta)) \subseteq (\ker Q^F(1/\beta) \cap \ker Q^F(-1/\beta)) . \quad (6.9)$$

For the other cases, the mixing of the Q^F and Q^B terms in the Toda potential prevents such a simple analysis.

In the next section we will consider the relevance of this discussion of duality in affine Toda theories for the results found in sections 3 and 4

7 Conserved quantities: from affine Toda field theories to perturbed conformal field theories

The discussion of duality in the previous two sections applies to the affine Toda field theories. However, the results that we would like to understand pertain to perturbed superconformal field theories. The relation between the two problems is quite simple: affine Toda conserved

densities are also conserved densities in the perturbed theories if they can be written in terms of the generators \hat{G} and \hat{L} of the unperturbed theory.

Actually, both conditions are related: the space $\hat{\mathcal{F}}$ can be characterised in terms of a cohomology involving precisely the same operators used to characterise the Toda conserved densities (although at potentially different values of β , clearly).

In the Virasoro case, the irreducible vacuum representation may be found as the BRST cohomology of the Fock space \hat{F}_0 where the BRST maps are of the form $Q_+^{(k)}$ or $\hat{Q}_+^{(k)}$, whichever is appropriate [18, 19]. Similarly, the action of a primary field is given as the action of a suitably screened vertex operator of the form (5.2).

We assume that these results transfer to the representations of the super-Virasoro algebra as well.

Our computations of the conserved quantities of the affine Toda theories (5.4) must then be altered to calculate the intersection of the kernels of the potentials corresponding to the perturbations of the super-Liouville action (5.1) with the BRST cohomology of the screening charge corresponding to the potential in the super-Liouville action.

Although there are exact formulations of the super Virasoro module $\hat{\mathcal{F}}$ in terms of the Fock module \hat{F}_0 as a cohomology space, it is only in rare cases that we are able at present to relate this to the kernels of the perturbing potentials. However, we can certainly be sure that the space $\hat{\mathcal{F}}$ has the same structure for a perturbation and its dual, since the superconformal generators of the unperturbed theory are identical in these cases. As a result, to compare the conserved densities of the perturbed superconformal models we need only compare the kernels of the perturbing potentials. Our results are presented below.

7.1 The perturbation $\hat{G}_{-1/2}\hat{\phi}_{1,3}$

Viewed as a perturbation of the superconformal model with $t = 1/\beta^2$, $\oint \hat{G}_{-1/2}\hat{\phi}_{1,3}$ has the form $Q^F(1/\beta)$. Consequently, we can say that the perturbations by $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ with $t = t_0$ and $t = 1/t_0$ have equal conserved densities when t_0 is irrational or 1.

For all other cases the free-field argument does not allow us to say whether they have the same conserved quantities or not, except when \hat{F}_0 is of type $\text{III}_-^0(+)$. In this case we have $\ker Q^F(1/\beta) \supset \ker Q^F(\beta)$ for $\beta^2 = 2k + 1$. This implies that the perturbations by $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ with $t = 3, 5, \dots$ may have more conserved densities than those with the dual values $t = 1/3, 1/5, \dots$. However, we do not observe such an enhancement in table 3.

7.2 The perturbations $\hat{\phi}_{1,3}$ and $\hat{\phi}_{1,5}$

The perturbations $\oint \hat{\phi}_{1,3}$ and $\oint \hat{\phi}_{1,5}$ viewed as perturbations of the superconformal model with $t = 1/\beta^2$ have the forms $Q^B(1/(\beta\sqrt{2}))$ and $Q^B(\sqrt{2}/\beta)$ respectively. As a result, the perturbations by $\hat{\phi}_{1,3}$ with $t = t_0$ and by $\hat{\phi}_{1,5}$ with $t = 1/t_0$ have equal conserved quantities for $t_0/2$ irrational or 1.

For all other cases, the free-field argument does not allow us to say whether these models related by duality have the same conserved quantities or not, except when \hat{F}_0 is of type $\text{III}^0_-(+)$, when $\ker Q^B(1/\sqrt{k}) \supset \ker Q^B(-\sqrt{k})$.

This is relevant when $Q^B(1/\sqrt{k}) = \oint \hat{\phi}_{1,3}$, i.e., $t = 2/k$ and when $Q^B(1/\sqrt{k}) = \oint \hat{\phi}_{1,5}$, i.e., $t = 1/(2k)$.

In the first case, the $\hat{\phi}_{1,3}$ perturbations at $t = 2/k$ will certainly have no fewer conserved densities than the perturbations by $\hat{\phi}_{1,5}$ at the dual values $t = k/2$, and may have more. We see that for the first value $t = 1$ that $\hat{\phi}_{1,3}$ does have more conserved quantities than $\hat{\phi}_{1,5}$.

Similarly in the second case, the $\hat{\phi}_{1,5}$ perturbed theories at $t = 1/2k$ may have more conserved densities than the $\hat{\phi}_{1,3}$ perturbed theories at the dual values $t = 2k$. This is indeed borne out, since the extra fermionic conserved quantities of $\hat{\phi}_{1,5}$ at $t = 1/(2k)$ have the same weight as $\hat{G}_{-1/2}$ acting on the extra vector u_2 in the kernel of $Q^B(1/\sqrt{k})$: since $\hat{\phi}_{1,5}|_{t=1/2k} = Q^B(1/\sqrt{k})$, we must use the value $t = 1/k$ to evaluate the weight of u_2 using (6.4). We find $h(u_2) = h_{0,3p-1} = h_{0,3k-1} = 2k - 1$, so that $h(u_2) + 1/2 = 2k - 1/2$ as required.

In all cases where t rational, it is also possible that the number of conserved quantities is increased over the number for generic irrational t . This may happen either because the intersection of the kernel of the perturbing potential with the polynomials in \hat{G} and \hat{L} increases, or because the dimension of the kernel itself jumps. This is a generic reason for the possibility of an increase in the number of conserved quantities, but in special cases we can provide a more detailed analysis which also allow us to count exactly the number of conserved quantities. These cases are treated in the next section.

8 Results on certain exceptional cases

8.1 Relation with $\widehat{su}(2)_2$

There is a construction of the affine algebra $su(2)$ at level 2 ($c = 3/2$) in terms of one free boson and one free fermion for which the currents $E(z)$ and $F(z)$ are given exactly by $Q^F(1)$ and $Q^F(-1)$. Consequently, the conserved currents for $C^{(2)}(0, 2)$ are the singlets of the zero-grade $su(2)$ subalgebra, for which the generating function is known. To count the number of nontrivial conserved currents it is only necessary to factor out the generating function of $su(2)$ singlets by the action of \hat{L}_{-1} . If the number of nontrivial conserved densities of the Toda theory at level n is d_n , we then have

$$\begin{aligned} \sum d_n q^n &= 1 + (1 - q) \left[\prod_{n=1}^{\infty} \left(\frac{1 + q^{n+1/2}}{1 - q^{n+1}} \right) - 1 \right] \\ &= 1 + q^{\frac{3}{2}} + q^2 + q^{\frac{7}{2}} + 2q^4 + q^{\frac{9}{2}} + 2q^{\frac{11}{2}} + 4q^6 + 2q^{\frac{13}{2}} + q^7 + 5q^{\frac{15}{2}} + \dots \quad (8.1) \end{aligned}$$

Since in this case \hat{F}_0 is of type III^{00}_- , the kernel of $Q^F(1)$ is also equal to the irreducible vacuum Virasoro representation. Therefore, all the conserved quantities found above are entirely expressed in terms of vacuum descendants, that is, in terms of $\hat{L}(z)$ and $\hat{G}(z)$, and

the number of conserved quantities of the $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ perturbation at $t = 1, c = 3/2$ is also given by the formula above. This counting agrees exactly with the results in table 3.⁸

8.2 Cases when $h = 1$

When the perturbing field $P(z)$ has conformal weight 1, the integral of $P(z)$ automatically commutes with all the modes \hat{L}_m . Consequently, every polynomial in \hat{L} and its derivatives is a conserved density, so that we have a lower bound on the number of nontrivial conserved densities given by

$$\begin{aligned} \sum d_n q^n &= 1 + (1 - q) \left[\prod_{n=1}^{\infty} \left(\frac{1}{1 - q^{n+1}} \right) - 1 \right] \\ &= 1 + q^2 + q^4 + 2q^6 + 3q^8 + q^9 + \dots \end{aligned} \quad (8.2)$$

The values of t for which each perturbation has weight 1 are $t = 1$ for $\hat{G}_{-1/2}\hat{\phi}_{1,3}$, $t = 3/2$ for $\hat{\phi}_{1,3}$ and $t = 2/3$ for $\hat{\phi}_{1,5}$. In these latter two cases we see that this exhausts the nontrivial conserved densities we have found.

8.3 $WB(0, m)$ algebras and $\hat{\phi}_{1,5}$ perturbations

The $WB(0, m)$ algebras (or fermionic WB_m algebras) are W-algebras with bosonic fields of spins $2, 4, \dots, 2m$ and one fermionic field of spin $m + 1/2$ [21, 22]. The $N = 1$ superconformal algebra is itself the $WB(0, 1)$ algebra. It is natural to ask whether the fermionic conserved quantities of the $\hat{\phi}_{1,5}$ perturbation of weight $2k - 1/2$, which occur only for $c = \hat{c}(2/4k)$, are related to the fermionic fields of the same weight of the $WB(0, 2k - 1)$ algebra.

The values of the central charge c corresponding to the $WB(0, m)$ minimal models are given by

$$c_m(p, p') = (m + 1/2) \left[1 - 2m(2m - 1) \frac{(p - p')^2}{pp'} \right], \quad (8.3)$$

where we require $p, p' \geq 2m - 1$. The minimal model representations are characterised by two weights of B_m ; the representation is in the Neveu-Schwarz / Ramond sector according to whether the difference of the two weights is in the root lattice or the root lattice shifted by

⁸ We stress that the above character codes the number of independent expressions built out of the modes \hat{L}_{-k} and $\hat{G}_{-m/2}$ at each level, modulo the singular vector $\hat{L}_{-1}|0\rangle$ and modulo total derivatives (i.e., modulo the left action of \hat{L}_{-1}). Here is another way of reaching the same conclusion. At $\beta = 1$, the free-field expressions for $\hat{L}(z)$ and $\hat{G}(z)$ are simply

$$\hat{L}_{\beta=1}(z) = -\frac{1}{2} [(\partial\phi)^2 : + : \psi\partial\psi :], \quad \hat{G}_{\beta=1}(z) = i\psi\partial\phi.$$

Both \hat{L}_1 and \hat{G}_1 are invariant under a simultaneous change of sign in ϕ and ψ . Since the second term of the potential of the $\hat{G}_{-1/2}\hat{\phi}_{1,3}$ perturbation can be obtained from the other (the screening charge) by this transformation, we conclude that any even differential polynomial in the free fields (hence any differential polynomial in \hat{L} and \hat{G}) is automatically conserved.

the spinor weight of B_m . There is also a restriction on the weights which is that their levels (as affine B_m representations) are equal to $p' - 2m + 1$ and $p - 2m + 1$ respectively.

The c values we are interested in are $\hat{c}(2/4k) = c_1(4k, 2)$. It is a simple check that

$$c_1(4k, 2) = c_{2k-1}(2k, 2k - 1) . \quad (8.4)$$

where we identify $c_1(4k, 2)$ with $c_1(2k, 1)$ in view of the constraint $p + p' = \text{even}$ ⁹. Although there are no minimal representations for this value of c , we can still consider the corresponding highest weights of $WB(0, 2k - 1)$. Let the $WB(0, m)$ highest weights be labelled by two B_m weights $[\lambda, \lambda']$ (finite parts of affine weights at respective level $p' - 2m + 1$ and $p' - 2m + 1$) whose conformal dimensions in the Neveu-Schwarz sector are given by

$$h[\lambda, \lambda'] = \frac{[p(\lambda + \rho) - p'(\lambda' + \rho)]^2 - \rho^2(p - p')^2}{2pp'} \quad (8.5)$$

with $\rho = \sum \omega_i$ (ω_i being the B_m fundamental weights). With $m = p' = 2k - 1$ and $p = 2k$, we find that

$$\begin{aligned} h[0, 0] &= 0 , & h[\omega_1, 0] &= 3/2 , \\ h[0, \omega_1] &= \hat{h}_{1,3} , & h[0, \omega_2] &= \hat{h}_{1,5} , & h[0, 2\omega_1] &= \hat{h}_{1,3} + 1/2 , \dots \end{aligned} \quad (8.6)$$

We can indeed see that the fermionic conserved quantities we find are exactly related to the fermionic fields of the $WB(0, 2k - 1)$ algebras. The three perturbations of the $WB(0, m)$ algebra by the fields $[0, \omega_1]$, $[0, \omega_2]$ and $[0, 2\omega_1]$ correspond to the affine Toda field theories $A^{(4)}(0, 2m)$, $A^{(2)}(0, 2m - 1)$ and $B^{(1)}(0, m)$ respectively.

The Zamolodchikov counting argument shows that the perturbation by $[0, \omega_2]$ does indeed have a fermionic conserved current of weight $m + 1/2$ and we identify this as our singular conserved quantity. This conserved quantity has also been investigated in more detail in [23].

9 Conclusions

In the superconformal case, as for the Virasoro and W minimal models, all integrable perturbations can be unravelled by the singular-vector argument. In fact, the situation is somewhat better in the superconformal case since, strictly speaking, the argument in the W case must be supplemented by duality (e.g., the Virasoro $\phi_{1,5}$ perturbed integrals of motion are never singular). In the superconformal case, even though the generic $\hat{\phi}_{1,5}$ perturbed integrals are never singular either, the integrability is revealed through an infinite sequence of non-generic singular fermionic conservation laws. Hence, the fermionic degree of freedom provides us a direct handle on the integrability of the $\hat{\phi}_{1,5}$ perturbation. Phrased differently, the presence of fermionic fields allows for conservation laws to be related to singular vectors not directly but through their square root, a possibility that is indeed realized in the $\hat{\phi}_{1,5}$ case.

⁹This is an example of a general level-rank duality $c_n(p + n, n) = c_p(p + n, n)$.

Notice that the singular-vector argument treats with equal simplicity standard perturbations as well as those not easily interpreted physically, such as the Ramond perturbations.¹⁰

Finally we would like to comment on the fact that duality, clearly broken in the $N = 1$ models, was apparently preserved in the bosonic models treated in [1]. Duality is indeed broken for the models in [1], but so far we have only observed it affine Toda theories; once we turn to Virasoro perturbed models and the spaces $\mathcal{F}_{r,s}$, duality is apparently restored.

As an example, consider the Sine-Gordon model whose potential is $Q^B(\beta) + Q^B(-\beta)$. For generic β , there are conserved densities of all positive even spins [3]. However, from explicit calculations, we find that for $\beta^2 = 1/2$ there appear extra conserved densities of weights $3, 5, 7, \dots$; for $\beta^2 = 1/3$ of weights $5, \dots$; for $\beta^2 = 2/3$ of weights $5, 7, \dots$. In each of these cases, the extra conserved quantity of lowest weight corresponds to the first additional vector in the kernel of $Q^B(\beta)$ (i.e., u_2). However, the additional extra conserved quantities are not directly related to the extra vectors identified in section 6.3 (i.e., $u_{i \geq 3}$); they are merely particular descendants of these vectors in the Fock module. However, it is clear many of these extra conserved densities will not survive when we consider the conserved densities in \mathcal{F} , for the simple reason that there are no quasi-primary states in \mathcal{F} of weight 3, 5 or 7 (i.e., there are no states in \mathcal{F} that are not total derivatives). Indeed, we have not found a single case where the number of conserved densities in \mathcal{F} for the Virasoro perturbation $\phi_{1,3}$ exceeds those for generic β .

The clarification of the duality breaking in Toda theories and its repercussion in the perturbed conformal models is certainly an interesting problem for future study.

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¹⁰ Note that while in the Ramond case, the singular-vector argument suggests the absence of an integrable perturbation, this does not constitute a proof of non-existence. However, explicit computer searches have yield no densities (other than the trivial case $\hat{h} = 1$) conserved for generic values of c .

A Non-integrable supersymmetric evolution equations with an infinite number of fermionic conservation laws

In this appendix, we present a neat example of a generic class of equations that display an infinite number of fermionic conservation laws but which are nonetheless non-integrable.¹¹ We first consider supersymmetric models. In a second step, we show that this curiosity is not an artifact of supersymmetry, by constructing a non-supersymmetric model with the same properties.

Consider the general class of space supersymmetric evolution equations:

$$R_t = [M(R)]_x + aRR_x - \left[\frac{(DR)(DM(R))}{R_x} \right]_x, \quad (\text{A.1})$$

where a is a constant and $M(R)$ is an even arbitrary differential polynomial in the superfield R :

$$R(x, \theta, t) = u(x, t) + \theta \sigma(x, t). \quad (\text{A.2})$$

Here u represents a commuting field, σ is an anticommuting field, θ is an anticommuting space variable and D is the super-derivative: $D = \theta \partial_x + \partial_\theta$, $D^2 = \partial_x$. R is thus an even superfield. The special structure of the equation readily implies the existence of an infinite number of conserved integrals, which are of the form $\int dx d\theta F(R)$ for any even function $F(R)$ which can be written as

$$F(R) = \frac{dG(R)}{dR} \equiv G(R)'. \quad (\text{A.3})$$

Indeed, using the evolution equation, one has

$$\begin{aligned} \int dx d\theta F_t &= \int dx d\theta \{ F'[M(R)]_x + aF'RR_x + F''(DR)(D[M(R)]) \} \\ &= \int dx d\theta \{ F'[M(R)]_x - aFR_x - F'[M(R)]_x \} \\ &= -a \int dx d\theta G_x = 0. \end{aligned} \quad (\text{A.4})$$

However, these conservation laws are somewhat trivial. For instance, with

$$F(R) = R^n = u^n + \theta n u^{n-1} \sigma, \quad (\text{A.5})$$

they read

$$\int dx d\theta F(R) = \int dx n u^{n-1} \sigma, \quad (\text{A.6})$$

and they disappear when $\sigma = 0$.

¹¹There is no paradox in this statement: integrability of a field theory requires not only an infinite number of conservation laws but also that these conservation laws be in involution.

The equation we have considered is a supersymmetric extension of

$$u_t = [M(u)]_x + auu_x , \quad (\text{A.7})$$

which is generically non-integrable. It is thus clear that for the type of nonlinearity considered here, the supersymmetric process itself is responsible for the infinite set of conservation laws.¹²

As a concrete example, consider the following supersymmetric extension of the Korteweg-de Vries (KdV) equation:

$$R_t = -R_{xxx} + 12RR_x + \left[\frac{(DR)(DR_{xx})}{R_x} \right]_x , \quad (\text{A.8})$$

whose component form reads

$$u_t = -u_{xxx} + 12uu_x + \left[\frac{\sigma\sigma_{xx}}{u_x} \right]_x , \quad (\text{A.9})$$

$$\sigma_t = 12(\sigma u)_x + \left[\frac{\sigma\sigma_x\sigma_{xx} - \sigma u_{xxx}u_x}{u_x^2} \right]_x . \quad (\text{A.10})$$

When $\sigma = 0$, this reduces to the KdV equation. Notice that this system differs from the two integrable fermionic extensions of the KdV equation [25] (in which cases σ has dimension 3/2 while it is 5/2 here).

The infinite family of conservation laws found above does not generalise the usual KdV conservation laws, whose leading term is of the form $\int dx (u^n + \dots)$. Conservation laws generalising the usual KdV ones involve non-local expressions in R , with leading term $R^n(D^{-1}R)$ (the formalism underlying the manipulations of such nonlocal charges is presented in [14]). It is a simple exercise to check that

$$\int dx d\theta (D^{-1}R), \quad \int dx d\theta R(D^{-1}R) , \quad (\text{A.11})$$

and

$$\int dx d\theta [R^2(D^{-1}R) + \frac{1}{4}R_x(DR)] , \quad (\text{A.12})$$

are conserved. For KdV type equations, these conservation laws are not remarkable: in a hydrodynamics context, they simply generalise the conservation of mass, momentum and energy. However, a nontrivial conservation law of the form

$$\int dx d\theta [R^3(D^{-1}R) + a_1R^2(DR_x) + a_2R_x^2(D^{-1}R) + a_3R_{xx}(DR_x)] , \quad (\text{A.13})$$

¹²Actually, the same result follows with weaker conditions, that is, without strict supersymmetry invariance – see below.

has not been found. For generalisations of the KdV equation, the non-existence of such a conservation law is a clear signal of non-integrability.¹³

With a simple deformation of the above supersymmetric KdV equation, we can easily construct a family of non-supersymmetric systems having an infinite number of fermionic conservation laws. One such deformation is:

$$u_t = -au_{xxx} + 12uu_x + b \left[\frac{\sigma\sigma_{xx}}{u_x} \right]_x, \quad (\text{A.16})$$

$$\sigma_t = 12(\sigma u)_x + \left[\frac{b\sigma\sigma_x\sigma_{xx} - a\sigma u_{xxx}u_x}{u_x^2} \right]_x. \quad (\text{A.17})$$

It is simple to check that for these equations the integrals $\int dx u^n \sigma$ are conserved for any value of a and b . However, the system is invariant under a supersymmetric transformation (defined by $\delta u = \epsilon\sigma$ and $\delta\sigma = \epsilon u_x$ with ϵ an anticommuting parameter) only for $a = b$.

We should stress that the fermionic character of the infinite sequence of trivial conservation laws that have been constructed here is rooted in the number of fermions (here one) in the system. If we consider instead the coupling of two fermions with two bosonic fields, then we could construct systems with an infinite number of *bosonic* conservation laws quite easily. Focusing on supersymmetric systems for definiteness, it is simple to verify (along the above argument) that for the following equation with $N = 2$ supersymmetry ($\mathcal{R} = u + \theta_1\sigma_1 + \theta_2\sigma_2 + \theta_1\theta_2w$, u, w bosonic and σ_i fermionic and $D_i = \partial_{\theta_i} + \theta_i\partial_x$)

$$\mathcal{R}_t = -\mathcal{R}_{xxx} + 12\mathcal{R}\mathcal{R}_x + \frac{1}{2} \left[\frac{\sum_{i=1,2} D_i \mathcal{R} (D_i \mathcal{R}_{xx})}{\mathcal{R}_x} \right]_x$$

the integrals $\int dx d\theta_1 d\theta_2 F(\mathcal{R})$ – which in the present context are bosonic – are conserved (with $F(\mathcal{R}) = dG/d\mathcal{R}$ for some G).

B The operator product of two NS fields for $N = 1$.

We are interested in the coefficients α and β in the operator products (with $|k\rangle \equiv \hat{\phi}_k(0)|0\rangle$):

$$\hat{\phi}_1(1) \hat{G}_{-1/2}|2\rangle = \hat{G}_{-1/2}|3\rangle + \left(\alpha \hat{L}_{-1} \hat{G}_{-1/2} + \beta \hat{G}_{-3/2} \right) |3\rangle + \dots, \quad (\text{B.1})$$

¹³It is amusing to notice that a travelling wave solution of the form $R = R(x - 12ct, \theta, t)$ of the above supersymmetric KdV equation is related to the super Weierstrass function. Simple manipulations (i.e., integrate the resulting equation once with respect to x , multiply the result by R_x , and integrate again) yields

$$R_x^2 - 4R^3 - 12cR^2 - 2(DR)(DR_x) + 2k_1R + k_2 = 0, \quad (\text{A.14})$$

where k_1, k_2 are integration constants. Now setting $P = R + c$, $g_2 = 2k_1 + 12c^2$ and $g_3 = k_2 - 2k_1c - 8c^3$, one can rewrite the above equation as

$$P_x^2 - 4P^3 + g_2P + g_3 - 2(DP)(DP_x) = 0, \quad (\text{A.15})$$

which is the defining equation for the super Weierstrass function [26]: $P = \wp(x; \tau) + \theta\delta\partial_\tau\wp(x; \tau)$, where \wp is the ordinary Weierstrass function, τ is the modulus and δ is the super modulus. $g_2(\tau)$ and $g_3(\tau)$ are the usual modular forms.

where $\hat{\phi}_i$ is a Neveu-Schwarz field of weight h_i , and where $\hat{\phi}_1$ is either fermionic ($\eta = 1$) or bosonic ($\eta = -1$). Then, using the standard relations

$$(\hat{L}_m - \hat{L}_{m-1}) \hat{\phi}_1(1) = \hat{\phi}_1(1) (\hat{L}_m - \hat{L}_{m-1} + h_1), \quad (\text{B.2})$$

$$(\hat{G}_m - \hat{G}_n) \hat{\phi}_1(1) = -\eta \hat{\phi}_1(1) (\hat{G}_m - \hat{G}_n), \quad (\text{B.3})$$

we find that

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \eta \begin{pmatrix} 2h_3(2h_3 + 1) & 4h_3 \\ 4h_3 & 2h_3 + 2c/3 \end{pmatrix}^{-1} \begin{pmatrix} (h_2 + h_3 - h_1)(h_3 + h_1 - h_2) \\ h_1 + h_2 - h_3 \end{pmatrix}. \quad (\text{B.4})$$

Inserting the values we use in section 2, and denoting p'/p by t ,

$$h_1 = \hat{h}_{r,1}, \quad h_2 = \hat{h}_{1,s}, \quad h_3 = \hat{h}_{r,s}, \quad (\text{B.5})$$

we find that in the two cases of interest, namely $(r, s) = (3, 5)$ and $(r, s) = (5, 3)$,

$$(\alpha \quad \beta) = \begin{cases} \left(\frac{\eta t(-1+3t)}{3(1-2t)(-1+t)^2}, \frac{\eta(5-15t+12t^2)}{3(-1+t)^2(-1+2t)} \right) & (r, s) = (3, 5), \\ \left(\frac{\eta(3-t)t}{3(-2+t)(-1+t)^2}, \frac{\eta t(-12+15t-5t^2)}{3(-2+t)(-1+t)^2} \right) & (r, s) = (5, 3). \end{cases} \quad (\text{B.6})$$

C The operator product of a R and a NS field for $N = 1$.

We are interested in the coefficients α, β, A, B in the operator products

$$\begin{aligned} \hat{\phi}_{NS}(1) |R\rangle &= |R'\rangle + \left(\alpha \hat{L}_{-1} + \beta \hat{G}_{-1} \right) |R'\rangle + \dots \\ \hat{G}_{-1/2} \hat{\phi}_{NS}(1) |R\rangle &= |R'\rangle + \left(A \hat{L}_{-1} + B \hat{G}_{-1} \right) |R'\rangle + \dots \end{aligned} \quad (\text{C.1})$$

where $\hat{\phi}_{NS}$ is a Neveu-Schwarz field of weight H , which is either fermionic ($\eta = 1$) or bosonic ($\eta = -1$), $|R\rangle$ is a Ramond highest-weight state of \hat{G}_0 eigenvalue λ and weight $h = \lambda^2 + c/24$, and $|R'\rangle$ is a Ramond highest-weight state of \hat{G}_0 eigenvalue λ' and weight $h' = \lambda'^2 + c/24$. Then, using the standard relations:

$$(\hat{L}_1 - \hat{L}_0) \hat{\phi}_{NS}(1) = \hat{\phi}_{NS}(1) (\hat{L}_1 - \hat{L}_0 + H), \quad (\text{C.2})$$

$$(\hat{G}_1 - \hat{G}_0) \hat{\phi}_{NS}(1) = -\eta \hat{\phi}_{NS}(1) (\hat{G}_1 - \hat{G}_0), \quad (\text{C.3})$$

$$\hat{G}_0 \hat{\phi}_{NS}(1) = \hat{G}_{-1/2} \hat{\phi}_{NS}(1) - \eta \hat{\phi}_{NS}(1) \hat{G}_0, \quad (\text{C.4})$$

we find that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \lambda' + \eta\lambda & 2 \\ 1/2 & -\lambda' + \eta\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2h' & 3\lambda'/2 \\ 3\lambda'/2 & 2h' + c/4 \end{pmatrix}^{-1} \begin{pmatrix} H + h' - h \\ \lambda' + \eta\lambda \end{pmatrix}. \quad (\text{C.5})$$

Inserting the values we use in section 2, and denoting p'/p by t ,

$$H = \hat{h}_{1,s} = \frac{(1-st)^2 - (1-t)^2}{8t}, \quad \lambda = \lambda_{r,1} = \frac{r-t}{\sqrt{8t}}, \quad \lambda' = -\eta\lambda_{r,s} = -\eta\frac{r-st}{\sqrt{8t}}, \quad (\text{C.6})$$

we find that

$$\beta = \eta \frac{(-1+s) t^{3/2}}{\sqrt{2} (-2+r+t-st) (1-r-2t+st)}, \quad (\text{C.7})$$

$$B = \frac{(-1+s) t (4-4r-3t+3st)}{4 (1-r-2t+st) (2-r-t+st)}. \quad (\text{C.8})$$

D Solutions to $\Delta\hat{h} = 3/2, 2, 5/2$

In these tables we take $(p-s)(r-1) \geq (p'-r)(s-1)$. The cases $(p-s)(r-1) < (p'-r)(s-1)$ may be obtained by taking $p' \leftrightarrow p$ and $r \leftrightarrow s$.

$(p-s)(r-1)$	+	$(p'-r)(s-1)$	+	$(r-1)(s-1)$	= 3	(r, s)	(p, p')	NS	R	Min.
3		0		0		$\begin{cases} (2, 1) \\ (4, 1) \end{cases}$	$\begin{pmatrix} (4, -) \\ (2, -) \end{pmatrix}$		\checkmark	\checkmark
2		1		0		No solutions			\checkmark	
2		0		1		(2, 2)	(2, 4)			
1		1		1		(2, 2)	(3, 3)			
1		0		2		(2, 3)	(4, 2)			
0		0		3		$\begin{cases} (2, 4) \\ (4, 2) \end{cases}$	$\begin{pmatrix} (4, 2) \\ (2, 4) \end{pmatrix}$			
$(p-s)(r-1)$	+	$(p'-r)(s-1)$	+	$(r-1)(s-1)$	= 4	(r, s)	(p, p')	NS	R	Min.
4		0		0		$\begin{cases} (2, 1) \\ (3, 1) \\ (5, 1) \end{cases}$	$\begin{pmatrix} (5, -) \\ (3, -) \\ (2, -) \end{pmatrix}$	\checkmark	\checkmark	\checkmark
3		1		0		No solutions				
3		0		1		(2, 2)	(5, 2)			
2		2		0		No solutions				
2		1		1		(2, 2)	(4, 3)			
2		0		2		$\begin{cases} (3, 2) \\ (2, 3) \end{cases}$	$\begin{pmatrix} (3, 3) \\ (5, 2) \end{pmatrix}$			
1		1		2		No solutions				
1		0		3		(2, 4)	(5, 2)			
0		0		4		$\begin{cases} (5, 2) \\ (3, 3) \\ (2, 5) \end{cases}$	$\begin{pmatrix} (2, 5) \\ (3, 3) \\ (5, 2) \end{pmatrix}$			
$(p-s)(r-1)$	+	$(p'-r)(s-1)$	+	$(r-1)(s-1)$	= 5	(r, s)	(p, p')	NS	R	Min.
5		0		0		$\begin{cases} (2, 1) \\ (6, 1) \end{cases}$	$\begin{pmatrix} (6, -) \\ (2, -) \end{pmatrix}$		\checkmark	\checkmark
4		1		0		No solutions			\checkmark	
4		0		1		(2, 2)	(5, 2)			
3		2		0		No solutions				
3		1		1		(2, 2)	(5, 3)	$\checkmark\checkmark$		\checkmark
3		0		2		(2, 3)	(6, 2)			
2		2		1		(2, 2)	(4, 4)			
2		1		2		(2, 3)	(3, 4)			
2		0		3		(2, 4)	(6, 2)			
1		1		3		No solutions				
1		0		4		(2, 5)	(6, 2)			
0		0		5		$\begin{cases} (2, 6) \\ (6, 2) \end{cases}$	$\begin{pmatrix} (6, 2) \\ (2, 6) \end{pmatrix}$			

Table 4: Solutions to $\Delta\hat{h} = 3/2, 2, 5/2$

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